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ON THE CAPITAL BUDGETING OF INTERRELATED PROJECTS

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I. INTRODUCTION

The literature on capital budgeting generally confines itself at best to a few of the relevant aspects of the investment decision problem, omitting certain others which are nonetheless essential ingredients of the problem which managers of firms must solve. Most, though by no means all, discussions of formal decision models assume certainty. Further, they make either of two extreme assumptions: that a meaningful investment demand schedule already exists, as in the manner of Fisher's transformation function,[1] or that all investment alternatives are independent in the sense that the acceptance of any set of them does not affect the feasibility or profitability of accepting any different set. With some notable exceptions [19, 16, 29 and 28] the literature also disregards the case of capital rationing, usually on the ground that rationing ought not to exist when firms behave rationally (in the narrow economic sense). A more cogent basis for reticence, although seldom expressed, is that the appropriate criterion for the choice of investments under rationing has not been agreed upon. Nevertheless, project selections are being made all the time, even without all the theoretical niceties having been resolved, although the need for such work is recognized.

The present paper is designed to survey the techniques available to the practitioner who must decide on an investment program consisting of a poten-

[1] See [10]. Fisher called it the "Opportunity Line." This is not the same as Keynes' Marginal Efficiency of Capital Schedule [17] or Dean's investment demand schedule in [7].

tially large number of interrelated projects, possibly also subject to constraints on capital or other resources, and who is willing to utilize the framework of certainty for making part of his analysis or who can adapt his problem to fit within the capabilities of relatively simple methods for dealing with random events. The basic outlook behind such a presentation would include the following points. First, short-run limitations on the availability of resources are a common experience. These may not be in the form of capital shortages, but rather on critical manpower or other inputs[2] which have to be accepted by the decision-maker. Second, although the theory of investment under uncertainty is not yet in an advanced state of development,[3] some progress has been made in providing aids to the decision-maker. Mathematical models are being developed by means of which the consequences of a series of complex assumptions can be followed to their conclusions, and optimization of specified objectives may be achieved. Directly related is the availability of efficient computational methods, usually employing computers, which enhances or even makes possible the application of formal methods. This combination of analysis and automatic calculation can then be utilized to obtain a "generalized sensitivity analysis" for the essential parameters of the problems. Thus a variety of intricate and sophisticated considerations can be brought to bear on capital budgeting decisions in such a way, that the talents and energies of management may be devoted to those aspects of the problem which may benefit most from informed and experienced judgement--the data inputs.

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[2] See [29] Sec. 7.2 and 7.5.

[3] Witness the problem of intertemporal comparisons and aggregation of utilities, which is reflected in part in the determination of the appropriate rate for discounting.

II. THE LORIE-SAVAGE PROBLEM

A. Integer Programming

A now familiar problem, first discussed by J. H. Lorie and L. J. Savage in 1955 [19] may serve as the point of departure for our discussion. Given the net present value of a set of independent investment alternatives, and given the required outlays for the projects in each of two time periods, find the subset of projects which maximizes the total net present value of the accepted ones while simultaneously satisfying a constraint on the outlays in each of the two periods. The problem may be generalized to an arbitrary number of time periods and stated formally as an integer programming problem, using the following notation. Let b_j be the net present value of project j, when discounting is done by the appropriate rate of interest;[4] let c_{tj} be the outlay required for the j^{th} project in the t^{th} period; let C_t be the maximum permissible expenditure in period t. We define x_j to be the fraction of project j accepted, and require that x_j be either zero or one. The model may then be written as,

II.1

a) Maximize $\sum_{j=1}^{n} b_j x_j$

b) Subject to $\sum_{j=1}^{n} c_{tj} x_j \leq C_t$, $\quad t = 1, \ldots, T$

c) $0 \leq x_j \leq 1$, $\quad j = 1, \ldots, n.$

d) x_j integral.

[4] This is usually referred to as the cost of capital.

Solution of this problem depends only upon finding a good integer programming code. Given that the $\underline{c}_{tj}$ are nonnegative the problem is bounded, and a finite optimum exists.[5] However, integer programming algorithms still perform unpredictably[6] and so it may be useful to mention alternative methods which will also be relevant for our later discussion. Before doing so it should be pointed out that the budgets expressed in constraints II.1b may refer to resources which are limited in supply in the short run, in addition or alternative to constraints on capital expenditures. An example might be drawn from a firm such as Sears, Roebuck and Company when it is planning a sharp increase in the number of stores. A limit on its expansion could come about from the availability of managers for the new stores, a number that cannot be increased simply by bidding managers away from competitors and immediately putting them in charge of Sears stores. A period of time is required for the new personnel to learn the ways of the organization, to make effective communication and execution of management policies possible. Such an interpretation of budgets probably carries more realism than the model of capital rationing implied by the original Lorie-Savage formulation [29, pp. 126-27].

A related point requires some amplification here. The example above suggests models in which the constraints also express generation or release of resources through the adoption of new projects. Thus a store might also produce future managers by training assistant managers for that duty. Basically the only change required in the formal model II.1 would be to allow the coefficients $\underline{c}_{tj}$ to be negative as well as nonnegative. However, such changes require

[5] See [29] Chap. 4.

[6] See [22]; also the author's experience with an All-Integer code, IPM3, was that a problem of this type with three constraints II.1b and 10 projects failed to converge within 5000 iterations. See also [12].

some more fundamental reinterpretations of the explicitly dynamic character of the process being modelled, and this falls outside the sphere of interest here. In addition, these problems have been treated at length in [29] Chapters 8 and 9.

B. The Linear Programming Solution

The Lorie-Savage problem may also be regarded as a simple linear programming problem by dropping the requirement II.1d that the $\underline{x}_j$ be integers. The major consequence from this change here is that some of the $\underline{x}_j^*$, the values of $\underline{x}_j$ in the optimal solution, may turn out to be proper fractions. Fortunately it is possible to prove that there is an upper limit on the number of fractional projects given by $\underline{T}$, the number of periods for which budget constraints exist.[7] The difficulties in the use and interpretation of the linear programming solutions may be summarized by citing the following points. First, the projects may not, in fact, be completely fixed in scale, or the budgets, designed primarily for control purposes, do permit a degree of flexibility. On the other hand, some projects are essentially discrete, as in location problems, and also the budgeted inputs may be rigidly limited. In addition, the maximum number of fractional projects increases when interrelationships between projects must aslo be taken into account. Our concern in this paper is exclusively with the problem involving discrete projects, and hence we shall put aside the linear programming solution.

Before going on to alternative approaches, however, it should be pointed out that Lorie and Savage proposed a trial and error method for finding the integer solution. When analyzed in the framework of mathematical programming it may be seen that they sought to solve the integer programming problem by finding (by trial and error) the optimal dual variables for the budget constraints of a linear programming formulation. Unfortunately, the

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[7]Proof of this and related propositions are given in [29] Sec. 3.8.

desired quantities do not always exist, and their cumbersome technique, which is barely manageable in the two-constraint case, fails to give any indication when the quest is in vain and should be abandoned. However, these difficulties are fully discussed in [29] and need not detain us here.[8]

C. Dynamic Programming

The problem as formulated in equations II.1 may be recognized as a special case of the knapsack or flyaway kit problem [6, 4, pp. 42-47] when the number of "budget" constraints is small. It arises, for example, when a camper must choose the number of each of n items he wishes to carry in his knapsack when the utility to him of each is given, and total volume and weight limitations are imposed by the size of the knapsack.[9] It may also be interpreted as the number of spare parts to be taken along by a submarine for which similar limitations on weight and volume exist, but for which the benefit of the spare-parts kit involves the probability that a part is needed and the cost of being without it. Leaving stochastic aspects for a later section of this paper, we may write down the dynamic programming formulation of this general problem and briefly discuss some shortcuts for the Lorie-Savage problem.

As in the dynamic programming solution to the knapsack problem, the time sequence is replaced by the sequence of projects being considered, and the ordering of the projects is arbitrary.[10] The method consists of determining the list of projects which would be accepted if the "budgets" in the T periods were $C_1', C_2', \ldots, C_T'$, and selection were restricted to the first i projects. This is

[8] The difficulties involve the duals to integer programming problems, see [29] Chap. 2, and Sections 3.5, 4.2, and 5.8.

[9] This differs from the problem stated in II.1 in that constraints II.1c are omitted.

[10] Some preliminary screening and rearranging may help to cut computation time.

done for $i=1,\ldots,n$, and within each "stage" i, for all feasible vectors, $C' = (C_1', C_2', \ldots, C_T')$, where feasibility means that $0 \leq C_t' \leq C_t$, $t=1,\ldots,T$. We define $f_i(C_1', C_2', \ldots, C_T')$ as the total value associated with an optimal choice among the first i projects when funds employed are as defined. The basic recurrence relationship then may be stated as

$$f_i(C_1', C_2', \ldots, C_T') = \max_{0 \leq x_i \leq 1} [b_i x_i + f_{i-1}(C_1' - c_{1i}x_i, C_2' - c_{2i}x_i, \ldots, C_T' - c_{Ti}x_i)] \qquad i = 1,\ldots,n \tag{II.2}$$

for $\qquad C_t' - c_{ti} \geq 0, \; t = 1,\ldots,T$

and $\qquad f_o\,(C') = 0$

where $f_i(C')$ is the total value of the optimally selected projects with projects $i + 1$, $i + 2, \ldots, n$ still to be considered and the unallocated funds are given by C'.

Two departures from the usual dynamic programming version of the knapsack problem may be noted. First, the number of "budget" constraints is an arbitrary number T, which may be significantly greater than two. Also, the x_j are either zero or one here, whereas in the knapsack problem they may be any integers usually up to some upper limit on each item. Dantzig regards even two budgets for the knapsack problem to be one too many,[11] and Bellman [3] suggests that a second constraint be handled by use of a Lagrangian multiplier by maximizing

$$\sum_{i=1}^{n} b_i x_i \;-\; \lambda \sum_{i=1}^{n} c_{2i} x_i \tag{II.3}$$

subject to the single constraint

$$\sum_{i=1}^{n} c_{1i} x_i \leq C_1. \tag{II.4}$$

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[11] "It [the dynamic programming approach to the knapsack problem] is recommended where there are only a few items and only one kind of limitation." [6] p. 275.

A value of $\underline{\lambda}$ is assumed and the one-dimensional recurrence relation

II.5 $$f_i(C_1') = \max_{0 \leq x_i \leq 1} [(b_i - c_{2i})x_i + f_{i-1}(C_1' - c_{1i}x_i)]$$

is solved as before. Bellman comments, "The value of $\underline{\lambda}$ is varied until the second constraint is satisfied. In practice, only a few tries are required to obtain the solution in this manner." [3, p.724] Of course, there is no guarantee that the second (or first) constraint will be satisfied exactly. Thus, depending on the numerical values of the parameters, the search for the optimal $\underline{\lambda}$ may not be quite as short as a "few tries."[12] Certainly, one would not want to resort to two Lagrangian multipliers to handle three kinds of limitations. The difficulties with such a procedure have already been referred to in connection with the Lorie-Savage solution to this problem.[13]

Joel Cord, in an article in Management Science [5], utilizes dynamic programming for project selection in just the way Bellman suggests, although Cord's constraints are on total outlay and average variance or return. We shall return to the substance of Cord's article below. For the moment, we merely wish to point out that the zero-one limitation on the unknowns makes it possible to handle both constraints efficiently without recourse to Lagrangian multipliers, and that even more can be handled when the number of projects to be considered is not loo large.[14] The computer program flow-charted in Appendix A makes use of this property, and by careful housekeeping[15] is able to handle a reasonable number of constraints.

[12] In part, this depends on what one regards as "few."

[13] See footnote 8.

[14] What constitutes a reasonable number of projects has not been determined, although computational problems for capital budgeting applications are apt not to be limiting. More significantly, the use of Lagrangian multipliers has led Cord to overlook the optimum solution in his numerical problem [5, p. 341]. The actual optimum consists of projects 5,8,12,13 and 20. Cord's erroneous solution appears to stem from his use of a fixed search interval for the Lagrangian multiplier. Other computational aspects are discussed in Appendix A.

[15] E.g., by making use of lexicographically ordered vectors.

III. INTERDEPENDENT PROJECTS WITHOUT BUDGET CONSTRAINTS

Difficulties with the usual "text-book" methods of capital expenditure evaluation, e.g., internal rate of return or present value, arise when the independence assumption between projects does not hold. How strong this assumption is may be seen when one considers that alternative to almost every project is the possibility of its postponement for one or more periods, with concomitant changes in outlays and payoffs. These, of course, form a mutually-exclusive set of alternatives since it would be deemed uneconomical, if not impossible, to carry out more than one of them.[16] Mutual exclusion is by no means the only alternative to independence, even though this is the only other possibility which is usually raised in the literature. Contingent or dependent projects can arise, for instance, when "acceptance of one proposal is dependent on acceptance of one or more other proposals " [29,p.11]. One simple example would be the purchase of an extra-long boom for a crane which would be of little value unless the crane itself were also purchased; the latter, however, may be justified on its own. When contingent projects are combined with their independent "prerequisites" we may call the combination a "compound project." Thus a compound project is characterised by the algebraic sum of the payoffs and costs of the component projects. It is of course true that in principle all contingent projects can be represented by sets of mutually-exclusive compound projects. In practice, however, this is likely to prove an undesirable way of handling them, for the number of compound projects may be very large. In the present section of this paper we shall take up the treatment of interdependent projects in the context of the models discussed in the previous section. We shall also take up an alternative formulation which allows a wider variety of interrelationships, omitting, for the present, consideration of budget constraints.[17]

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[16] See, for example, Marglin's analysis of the "Myopia Rule" in [20].

[17] In this paper we omit consideration of interdependence between existing projects and new ones. Models which deal with such problems will be of the type referred to at the end of Sec.IIA, although the tools developed here will still be applicable.

A. Linear and Integer Programming

The methods of handling interrelationships of the types mentioned in the paragraph above were discussed at length elsewhere, and hence a brief summary here will suffice.[18]

Consider a set J of mutually-exclusive projects from which at most one is to be selected. This constraint may be expressed by

III.1 $$\sum_{j \in J} x_j \leq 1.$$

With the implied nonnegativity constraint on the x_j, this has the effect of limiting the sum of projects accepted from the set J to a single one. When an integer programming algorithm is utilized to solve the problem one is assured that at most one of the x_j equals unity, $j \in J$, while the remainder from the set are zero. Solution by linear programming leads to the possibility that several projects from the set J will be fractionally accepted. The total number of such projects will still be limited although a situation with fractional x_j^* for more than one project in the set J can arise.[19] Note that the unity upper bound constraint on the x_j individually, $j \in J$, are now superfluous.

Contingent projects may be handled in a similarly simple manner. If project r may be undertaken only if project s is accepted, but project s is an independent alternative, then we may express the relationship by

III.2 a) $$x_r \leq x_s$$

and b) $$x_s \leq 1.$$

Thus, if $x_s^* = 1$, i.e., it is accepted in the optimal solution, then $x_r \leq 1$ is the effective constraint. Otherwise, $x_r \leq 0$, together with the nonnegativity require-

[18] See [29] pp.10-11, 32-43, and the analysis of the duals to these constraints, pp. 147-52.

[19] See [29] p.37. For an interpretation of this result, see also [29]p. 32.

ment, forces $\underline{x}_r^* = 0$. If projects u and v are mutually-exclusive alternatives, and project r is dependent on acceptance of either project u or project v, this interrelationship may be expressed by

$$\text{III.3} \qquad \begin{array}{ll} \text{a)} & x_u + x_v \leq 1 \\ \text{b)} & x_r \leq x_u + x_v \end{array}$$

Hence, if one of the pair, u and v is accepted, then constraint III.3b becomes $\underline{x}_r \leq 1$. If neither u nor v is accepted, then III.3b becomes $x_r \leq 0$, implying, once again, that $\underline{x}_r^* = 0$. Similarly, if projects r and s are mutually-exclusive and dependant on the acceptance of either project u or v, two mutually-exclusive alternatives, the interdependence may be represented by the two constraints

$$\text{III.4} \qquad \begin{array}{ll} \text{a)} & x_u + x_v \leq 1 \\ \text{b)} & x_r + x_s \leq x_u + x_v. \end{array}$$

Contingent chains can easily be built up, as when acceptance of project r is dependent on acceptance of project s, which in turn is dependent on acceptance of project u:

$$\text{III.5} \qquad \begin{array}{ll} \text{a)} & x_u \leq 1 \\ \text{b)} & x_s \leq x_u \\ \text{c)} & x_r \leq x_s, \end{array}$$

etc.

B. Dynamic Programming

A glance at inequality III.1 which states the restriction on mutually-exculsive projects reveals that its algebraic form is exactly that of the budget constraints, restrictions II.1b, i.e., the coefficients are all nonnegative. This was the only requirement necessary for applying the knapsack problem formulation,

and hence solution by dynamic programming hinges only on the computational problem derived from having additional constraints. Since at most a few of the projects will be mutually-exclusive at one time, although there may be many such sets, the number of nonzero coefficients will be small. This has the effect of speeding up the calculations and keeping the "in-lists"--the lists of accepted projects--relatively small.

In principle, contingent projects can be handled as sets of mutually-exclusive compound alternatives, as was pointed out above. The difficulty with the dependency relation III.2a may be seen by putting both unknowns on the same side of the inequality, as in

III.6 $$x_r - x_s \leq 0.$$

The zero right side presents no difficulty. However, the negative coefficient of $\underline{x}_s$ does. A solution with $\underline{x}_s^* = 1$, $\underline{x}_r^* = 0$ would be feasible; yet, in the dynamic programming calculation, the unallocated amounts for each of the budgets must be nonnegative. The algorithm tests for this condition at each step.

A way out of the difficulty is to multiply the constraint by -1:

III.7 $$x_s - x_r \geq 0.$$

In this form the feasibility test of the dynamic programming calculation is satisfied only when the dependency condition is satisfied as intended. In addition, $\underline{x}_s^* = 1$, $\underline{x}_r^* = 0$ is also feasible in both senses. To make the constraint have the proper form we also put on the "budget" of unity,

III.8 $$0 \leq x_s - x_r \leq 1$$

and add the additional "feasibility test" for this (say the $\underline{k}^{th}$) budget,

III.9 $$C_k' - \sum c_{ki}x_i \leq 1.$$

The more complex interrelationship given by restrictions III.4a and b can also be represented by the constraints,

III.10
$$\text{a)} \quad 0 \leq x_u + x_v - x_r - x_s \leq 1$$
$$\text{b)} \quad x_u + x_v \leq 1,$$

with the additional feasibility test III.9.[19a]

C. <u>Quadratic Integer Programming and the Generalization of Second-order Effects</u>

Although contrary to the teaching of the principle of "Occam's Razor," it is possible to represent the above interdependencies by means of quadratic constraints superimposed on the 0/1 integer requirement for the x_j. For two mutually-exclusive projects, r and s, the relevant restriction, in addition to nonegativity on the unknowns, would be

III.11 $$x_r \cdot x_s \leq 0$$

which makes either $x_r^* = 1$ or $x_s^* = 1$, but not both $x_r^* = 1$ and $x_s^* = 1$. For the dependence of project r on project s, we require

III.12 $$x_r(1-x_s) \leq 0.$$

Thus, if $x_s^* = 1$, x_r^* may be either zero or one and the restriction will be satisfied. However, if $x_s^* = 0$ then $x_r^* = 0$ necessarily.

A generalization to include all pair-wise of second-order effects has been offered by S. Reiter [24]. A triangular payoff matrix

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[19a] The method actually employed in a number of numerical problems, and which made use of some special features of the program, required only that projects be pre-ordered so that the independent members of a set always preceded the dependent ones. Such sequencing obviated use of a second constraint for any given dependence relation.

$$\text{III.13} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & b_{nn} \end{pmatrix}$$

is defined for the set of n investment alternatives such that the payoff (e.g., net present value) from the acceptance of project r alone is b_{rr}, and the additional payoff from acceptance of both projects r and s is b_{rs}, apart from the payoff from acceptance of project r, b_{rr}, and project s, b_{ss}. An optimal partitioning of the set of project indices, $i=1,\ldots,m$ into two mutually-exclusive and exhaustive subsets, $\{i_1,\ldots,i_r\}$ and its complement is sought such that the total payoff from the first subset, the "in-list," is maximum. Given any such partitioning the payoff Γ_α associated with an in-list α may be thought of being obtained by crossing out all rows and columns of B which are not on the in-list and adding up the b_{ij} that remain. In Reiter's context such a payoff matrix represents as optmization problem only if some b_{ij} are negative, for otherwise the optimal in list α^* would be the entire list of projects. However, in some of our extentions, below, this need not be the case. It is also clear that the elements below the diagonal are not needed to represent any of the interaction effects which can be handled in an $n \times n$ array.

A few examples paralleling our earlier discussion will bring out some of the features of this development. If projects r and s are mutually-exclusive, all that is necessary to prevent their simultaneous adoption is to make b_{rs} a highly negative penalty.[20] Of course, all that is required here is the value

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[20] This is analogous to the penalty for keeping artificial vectors out of the optimal basis of a linear programming solution. See [4b] Section I.4.

b_{rs} which makes the total $b_{rr} + b_{ss} + b_{rs}$ represent the net value of adopting both alternatives, a quantity which will certainly be smaller than either b_{rr} or b_{ss}, the value of either alternative alone. Hence, b_{rs} will be negative. Using the penalty of $-M$ we may represent a set of mutually-exclusive projects, for purposes of illustration assumed to be the first k projects, by

III.14 $$B = \begin{pmatrix} b_{11} & -M & -M & \cdots & -M \\ 0 & b_{22} & -M & \cdots & -M \\ 0 & 0 & b_{33} & \cdots & -M \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & b_{kk} \end{pmatrix}$$

A dependence of project r on project s is handled by letting b_{rr} be the cost, a negative quantity while b_{rs} represents the benefit from having r in addition to s. b_{ss} would be the net payoff from accepting project s alone. We may finally illustrate this formulation with the situation expressed by equations III.4a and b, in which projects u and v are mutually-exclusive, as are projects r and s, and in addition to which acceptance of project r or s is dependent on acceptance of project u or v.

III.15 $$B = \begin{pmatrix} -b_r & -M & b_{ru} & b_{rv} \\ 0 & -b_s & b_{su} & b_{sv} \\ 0 & 0 & b_{uu} & -M \\ 0 & 0 & 0 & b_{vv} \end{pmatrix}$$

Here the quantities $-b_r$ and $-b_s$ represent the cost of projects r and s, respectively; b_{uu} and b_{vv} represent the net benefit from doing project u or v alone; and b_{ru}, b_{su}, b_{rv}, b_{sv} represent the additional benefits from selecting both projects r and u, s and u, r and v, and s and v, respectively. Once again the quantity $-M$ is a large and negative penalty, intended to dominate its row and column by a substantial amount.

Before presenting an outline of Reiter's method for finding the optimal partitioning of the project indices, we may show that his problem is actually an integer quadratic programming problem:

III.16 Maximize $$XBX^t = \sum_{i=1}^{n} b_{ii}x_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} b_{ij}x_ix_j$$

where $\underline{X} = (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n)$, $\underline{X}^t$ is its transpose, and $\underline{x}_i = 0$ or 1. Thus the payoff $\underline{b}_{ij}$ is realized only if both $\underline{x}_i^* = 1$ and $\underline{x}_j^* = 1$. Otherwise the product $\underline{b}_{ij}\underline{x}_i^*\underline{x}_j^* = 0$. Similarly, since the $\underline{x}_i$ are restricted to zero and unity it is unnecessary to distinguish between $\underline{x}_i$ and $\underline{x}_i^2$. We shall return to other problems in which this formulation is of utility later on.

Reiter's method for maximizing III.16 is not algorithmic in the usual sense. It locates a local maximum (guaranteed to exist by the finiteness of the number of projects) by a gradient method that traverses "connected in-lists." By starting at random in-lists it generates a variety of local optima which can be arranged in ascending order. Whether the global optimum is reached depends on whether an initial in-list is selected which leads to the global optimum as its local optimum. Hence, it is possible to estimate probabilistically the chances that the global optimum will be reached. Optimal stopping rules (based on the value of improvement vs. the computational cost of obtaining it) and optimal fixed sample-size plans have also been developed [25 and 26].

The method may be described as follows. Given an arbitrary in-list $\underline{\alpha}$, compute the corresponding payoff $\underline{\Gamma}_{\alpha}$. A connected in-list $\underline{\alpha}'$ is one which differs from $\underline{\alpha}$ either by including one project not contained in $\underline{\alpha}$, or by excluding one project which is included in $\underline{\alpha}$. The quantity $\underline{\Gamma}_{\alpha}$ is computed for each $\underline{\alpha}'$ connec-

ted to $\underline{\alpha}$, and so is the gradient $\underline{G}_{\underline{\alpha}'} = \underline{\Gamma}_{\underline{\alpha}'} - \underline{\Gamma}_{\underline{\alpha}}$. The $\underline{\alpha}'$ corresponding to the largest $\underline{G}_{\underline{\alpha}'}$ is selected as the starting point for the next iteration and these are continued until no $\underline{\alpha}'$ can be found for which $\underline{G}_{\underline{\alpha}'} > 0$. Once this point has been reached the in-list with a local maximum has been found. (In the event of ties between $\underline{G}_{\underline{\alpha}'}$ along the way, a simple rule such as choosing that $\underline{\alpha}'$ with the smallest first index [where a difference exists] can be used to break the tie.)

Having found a local optimum one seeks to choose a new starting in-list which has not already been evaluated, so as to find another local optimum. The connected in-lists constitute branches on a tree which has many starting points that lead to the top (the local maximum). The "broader" the tree having the global optimum as its local maximum, the more likely it is that the global optimum will be found. The method guarantees only that the global maximum will be found with "probability one." Unfortunately, it will not be recognized as such short of evaluating every possible in-list along the way. However, by careful use of prior information [26] and aspiration levels to indicate when the procedure should be stopped, one can arrive at "good" programs rapidly.[21] A numerical example with its tree-structure is presented in Appendix B.

One final note regarding this approach to the selection of interdependent projects is required here. Although Reiter restricts himself to a consideration of second-order interactions exclusively, there is nothing in the method which requires this. A generalization to $\underline{k}^{th}$ order effects requires computations involving an

$$\overbrace{n \times n \times \cdots \times n}^{k}$$

array, for which specifying the numerical values of the parameters will be more difficult than obtaining a good solution. Increasing the dimensionality of the

[21] In any case, one would partition the matrix $\underline{B}$ into independent submatrices, if such exist, and use the method on the submatrices separately.

array does not change anything essential in the structure of the problem--the connectedness or finiteness of the iterative process for finding a local optimum--and hence only the computational problems are affected, though perhaps drastically.

IV. INTERDEPENDENT PROJECTS WITH BUDGET CONSTRAINTS

The methods outlined in the previous section, in addition to handling the interrelationships between projects, permit inclusion of budget constraints with varying degrees of difficulty. For linear and integer linear programming formulations the interrelationships are essentially like budgets, and this holds approximately for the dynamic programming formulation, as was discussed in Section III. The formulation as an integer quadratic programming problem also allows the introduction of linear restrictions such as the budget constraints, although progress toward an algorithm for such problems has not been rapid to date.[18].

The only point which bears a brief discussion in this section is the introduction of side-conditions of the budget type into the Reiter format. To accomplish this end with a single restriction one may simply utilize a Lagrangian multiplier as did Bellman in the knapsack problem. That is, the terms $\underline{c_j}$ from the constraint

IV.1
$$\sum_{j=1}^{n} c_j x_j \leq C$$

are introduced into the matrix B of III.13 with the Lagrangian $\underline{\lambda}$:

IV.2
$$B = \begin{pmatrix} b_{11} - \lambda c_1 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} - \lambda c_2 & b_{23} & \cdots & b_{2n} \\ & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & b_{nn} - \lambda c_n \end{pmatrix}$$

and the optimization procedure is repeated with varying values of $\underline{\lambda}$ until restriction IV.1 is met. This procedure is not without pitfalls. Suppose $\underline{\alpha}^{o}$ ($\underline{\lambda}$) is the in-list accepted as the best obtainable for the given aspiration level (stopping rule) and the given value of $\underline{\lambda}$. Suppose, further, that $\underline{\alpha}^{*}$ ($\underline{\lambda}$) is the optimal in-list for this value of $\underline{\lambda}$. The difference between $\Gamma_{\underline{\alpha}^{o}}$ ($\underline{\lambda}$) and $\Gamma_{\underline{\alpha}^{*}}$ ($\underline{\lambda}$) may arise (a) because larger $\underline{b}_{ij}$ are available, or (b) smaller $\underline{c}_j$ are available, or (c) both.[22] Only in case (a) are the constrained quantities $\underline{c}_j$ not involved. In both other instances, the most likely case being (c), this difference implies that the global optimum (for the given value of $\underline{\lambda}$) will show a greater amount of slack or smaller surplus in constraint IV.1 than does $\underline{\alpha}^{o}$ ($\underline{\lambda}$). A reasonable procedure for changing $\underline{\lambda}$ in the direction indicated by the presence of slack or violation of the restriction would be to begin the computations with the previous in-list $\underline{\alpha}^{o}$. The grid for such changes in $\underline{\lambda}$ should reflect the bias in the estimation of the optimal value of the Lagrangian multiplier.[23] It should be born in mind, nevertheless, that feasible solutions, which satisfy the constraint IV.1, will always be found (assuming that the constraint is consistent with the interrelationships) even though it is not guaranteed to be a global optimum.

One final observation which will be followed by a detailed discussion of a particular application below concludes this section. Given the discrete

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[22] The $\underline{c}_j$ are nonnegative quantities, and are multiplied by the positive constant and then subtracted from the $\underline{b}_{ij}$.

[23] More complete ways of handling this problem are being investigated.

optimization technique of Reiter the cost structure may also express the whole range of second-order interactions, generalizing to the following payoff matrix:[24]

$$\text{IV.3}\quad B = \begin{pmatrix} b_{11} - \lambda c_{11} & b_{12} - \lambda c_{12} & b_{13} - \lambda c_{13} & \cdots & b_{1n} - \lambda c_{1n} \\ 0 & b_{22} - \lambda c_{22} & b_{23} - \lambda c_{23} & \cdots & b_{2n} - \lambda c_{2n} \\ 0 & 0 & b_{33} - \lambda c_{33} & \cdots & b_{3n} - \lambda c_{3n} \\ & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & b_{nn} - \lambda c_{nn} \end{pmatrix}$$

V. PROBABILISTIC CONSIDERATIONS

Introduction of probability distributions into the models considered earlier cannot automatically be regarded as dealing with the problems of uncertainty. In the selection of projects whose outcomes are stochastic, it is not, in general, clear what the random variables are, or how they are distributed, although some attempts at the organization of data have recently been proposed.[25] More important, it is not yet clear how such outcomes should be evaluated.[26] We shall wave these matters aside in order to pursue our original aim of clarifying and expanding on simple methods of analyzing the consequences from given assumptions, hopefully useful considering the current state of the art.

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[24] Should the payoff matrix be partitionable into submatrices, as e.g., in

$$B = \left(\begin{array}{cc|cccc} b_{11} - \lambda c_{11} & b_{12} - \lambda c_{12} & 0 & 0 & \cdots & 0 \\ 0 & b_{22} - \lambda c_{22} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & b_{33} - \lambda c_{33} & & \cdots & \\ & & & \cdots & \cdots & \end{array}\right)$$

the problem may be restated as

$$\text{Maximize} \qquad Z(\ ) = \sum_{k=1}^{s} (XBX^t)_k$$

where each $(XBX^t)_k$ is a similar subproblem. The value of λ remains the same for al subproblems at any state since the budget constraint applies to all simultaneously.

[25] See, for example, [13] and [14].

[26] This problem remains conspicuously unsolved in [13]. However, see also [15].

A. Independent Investment Projects

Even investment alternatives which are independent in the physical sense but which have probabilistic payoffs introduce a hierarchy of difficulties in the selection of an optimal set. The foremost of these is the choice of criterion function for optimization.[27] Confining ourselves, as before, to economic benefits, we first look at expected value maximization. Under this criterion, and assuming that the decision-maker is satisfied that he has defined meaningful random variables and that he knows the shapes and parameters of their distributions, we distinguish between situations where the payoffs are independently distributed and where they are not. For the moment we consider the quantities subject to budget constraints to be certain.

With independently distributed outcomes the problem has not been altered in any essential way. The form of the original Lorie-Savage problem still applies, expected payoffs being substituted for certain ones. Integer programming or dynamic programming may be used to solve the problem, as before, or linear programming may be applied if its limitations are not important here. It should be obvious that most of the problems of uncertainty have been assumed away, although a model of this type of uncertainty has been used [2]. If the outcomes of investments are jointly distributed nothing new is introduced under expected value maximization. The same is not the case if the conditional distribution of outcomes of a particular project, given that another is undertaken, is different from the unconditional distribution. Such is the case, for example, if the profitability of a manufacturing facility is affected by the decision to build a warehouse nearby. Such second-order effects may be handled by quadratic integer programming, as in Section III.C, and solved by Reiter's method or our extension provided that

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[27] We are not concerned with a general discussion of utility here.

no more than a single constraint is imposed. The off-diagonal elements of the payoff matrix B would then express the expected value of the joint adoption of two projects while the diagonal elements are expected payoffs from acceptance of the projects by themselves. The costs would be subtracted from the diagonal elements, as in matrix IV.2.

B. Nonlinear Utility Functions

When the stochastic nature of the outcomes is given prominence in the problem, consideration must also be given to the form of the function whose optimization is being sought. Questions of risk aversion or risk preference lead to inclusion of quantities other than expected payoffs into the criterion or utility function. The meanings of the terms "risk aversion" and "risk preference" have been sharpened considerably in a recent paper by Pratt [23], a few of whose conclusions will be referred to below. This subject, which is intertwined with the concept of uncertainty, raises such additional issues as whose utility should be optimized, as well as the effect of expectations concerning the availability of future prospects on the current decision. The latter difficulty has already been relegated to another time by omitting consideration here of sequential decision procedures. The former we leave to others, assuming that a determination can be made by the decision-maker.

As Markowitz has pointed out [21, 9] the step to the simplest nonlinear utility function can be taken directly or indirectly. First one may assume that the utility function is quadratic. Unfortunately, the quadratic is not risk-averse in Pratt's sense.[28] However, if all outcomes are normally distributed

[28] One consequence of the quadratic utility function is that the "risk premium"--that additional amount which makes a fair gamble acceptable to a risk-averse gambler--increases with his initial wealth.

only the mean and variance of the total payoff enter into the utility function, reducing the utility function to a quadratic. We assume first that the outcomes are jointly normally distributed, but that the distributions are not contingent on the decisions themselves as in the preceding section. The resulting problem has a form exactly analogous to the Markowitz Portfolio Selection problem with the following exceptions. First, the projects are discrete, unlike the portfolio problem in which for a given security there are constant returns to scale. As a result, the problem is not to determine the proportion of the portfolio to be allocated to each eligible security, but to determine the list of projects to be accepted. Formally, this problem may be written as

V.1

$$\text{Maximize} \quad \mu - \lambda\sigma^2 = \sum_{i=1}^{n} \mu_i x_i - \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j$$

$$\text{subject to} \quad x_i = \begin{cases} 0 \\ 1 \end{cases}, \; i=1,\ldots,n$$

where μ is the expected value of the payoff of the accepted projects and σ^2 is the variance of the total payoff. The quantity λ here has an interpretation, being a measure of risk aversion--the rate of trade-off between reduction in expected value for reduction in variance. Thus μ is the sum of the expected payoffs μ_i from the individual projects which are accepted. The σ_{ij} are the covariances between the outcomes of projects i and j, $i \neq j$, and $\sigma_{ii} = \sigma_i^2$ are the variances. Since we have restricted the values of x_i to zero or one, $x_i = x_i^2$, and functional V.1 has a form similar to one treated before by Reiter's Discrete Optimization Method:

V.2
$$B = \begin{pmatrix} \mu_1 - \lambda\sigma_1^2 & -2\lambda\sigma_{12} & -2\lambda\sigma_{13} & \cdots & -2\lambda\sigma_{1n} \\ 0 & \mu_2 - \lambda\sigma_2^2 & -2\lambda\sigma_{23} & \cdots & -2\lambda\sigma_{2n} \\ 0 & 0 & \mu_3 - \lambda\sigma_2^3 & \cdots & -2\lambda\sigma_{3n} \\ & & \cdots & \cdots & \\ 0 & 0 & 0 & \cdots & \mu_n - \lambda\sigma_n^2 \end{pmatrix}$$

We may contrast this formulation with the one recently suggested by J. Cord which was referred to earlier. Cord considers the problem of "optimally selecting capital investments with uncertain returns, under conditions of limited funds and a constraint on the maximum average variance allowed in the final investment package " [5, p. 335]. Although Cord's figure of merit is "interest rate of return" of a project which is multiplied by the required outlay for the project,[29] nothing essential is changed by substituting the expected present value for this quantity. Similarly, the variance of the present value may be used in place of the variance he uses.[30] Cord solves this problem by dynamic programming utilizing the Lagrangian multiplier technique to obtain a solution which satisfies the constraint on variance. Using the notation of the preceding paragraph, we may restate his recurrence relation as

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[29] In his discussion, Cord indicates that by "interest rate of return" he means internal rate of return--that rate which equates the discounted value of inflows to the (assumed by Cord, sole) outflow. For this he was properly criticized by T.R. Dyckman [8] because of the effect of project lives on total benefit. Actually, Cord's use of this quantity is not consistent with application of the internal rate of return, which he multiplies by the outlay to obtain the figure of merit used in the objective function. His model can be made consistent, however, simply by interpreting "interest rate of return" as the uniform perpetual rate of return.

[30] Cord introduces a slight error into his solution by his weighting of the variance of returns when not all available funds are allocated. The weight $\underline{c_i}/\underline{C}$ in our notation is then too small, and as a result, the constraint on the average variance may be violated.

V.3 $$f_i(C') = \max_{\substack{0 \leq x_i \leq 1 \\ 0 \leq C' \leq C}} [\mu_i x_i - \lambda\sigma_i^2 x_i + f_{i-1}(C' - c_i x_i)]$$

where C' is the unallocated budget and f_i the total value of the selected projects when C' remains to be allocated and $n - i$ projects have been considered.

Cord explicitly assumes independence in the statistical sense between the project payoffs. Hence his total variance is the sum of the variances of the accepted projects, avoiding the quadratic terms and covariances. Although it simplifies the computations, this assumption, which he has difficulty in justifying, is not strictly necessary for application of dynamic programming. Having consciously patterned his approach after Markowitz he might also have considered Sharpe's Diagonal Model [26a] which is designed to simplify the computations for obtaining optimal portfolios by allowing only a restricted covariance between the eligible securities.

Specifically, Sharpe assumes that the returns on securities are related only through a relationship with some common factor, e.g., an index of general activity. If the random variable, the return on the i^{th} security, is denoted by p_i, and the level of the index is I, he expresses this return by

V.4 $$p_i = \mu_i + \beta_i I + w_i$$

where u_i and β_i are parameters, and w_i is a random variable with mean of zero and variance of σ_i^2. Further, the level of I is given by the sum of a systematic component, μ_I, and a random part, w_I, with mean of zero and variance of σ_I^2,

V.5 $$I = \mu_I + w_I.$$

Putting these relations together, with the assumption that Cov (w_i, w_j) = 0 he obtains

V.6
$$\text{Exp }(p_i) = \mu_i + \beta_i(\mu_I)$$
$$\text{Var }(p_i) = \sigma_i^2 + \beta_i^2\sigma_I^2$$
$$\text{Cov }(p_i,p_j) = \beta_i\beta_j\sigma_I^2$$

In the portfolio problem the unknowns are the proportions of the total amount invested which are allocated to various securities. By contrast, our variables are integer-valued; x_i = 0 or 1. Hence we may maximize total expected investment program payoff μ subject to an expenditure ceiling of C and a variance limit of σ^2, or alternatively, maximize payoff less $\lambda\sigma^2$ subject to the budget constraint, where

V.7
$$\mu = \sum_{i=1}^{n}(\mu_i + \beta_i\mu_I)\, x_i$$

V.8
$$\sigma^2 = \sigma_I^2 \left(\sum_{i=1}^{n} \beta_i x_i\right)^2 + \sum_{i=1}^{n}\sigma_i^2 x_i .$$

The recurrence relation may then be written as

V.9
$$f_i(C') = \max_{\substack{0\leq x_i\leq 1 \\ 0\leq C'\leq C}} \left[\mu_i x_i - \lambda x_i \left\{\sigma_i^2 + \sigma_I^2\, (\beta_i^2 + 2\beta_i \sum_{j=1}^{i-1}\beta_j x_j)\right\} + f_{i-1}(C' - c_i x_i)\right]$$

where the term enclosed in braces results from the factoring

V.10
$$\sigma^2 = \sigma_I^2 \left(\sum_{j=1}^{i} \beta_j x_j\right)^2 + \sum_{j=1}^{i}\sigma_j^2 x_j$$
$$= \sigma_I^2 \left(\sum_{j=1}^{i-1} \beta_j x_j\right)^2 + \sum_{j=1}^{i-1}\sigma_j^2 x_j + \sigma_I^2 x_i(\beta_i^2 + 2\beta_i \sum_{j=1}^{i-1} \beta_j x_j) + \sigma_i^2 x_i .$$

Contrary to Cord's viewpoint, the Lagrangian multiplier formulation above has the advantage here of tracing out a variety of efficient investment programs in the Markowitz sense[31] instead of requiring the decision maker to specify his tolerance for variance abstractly.

Utilization of an index to capture the major effect of covariation seems more appropriate in our context than in portfolio selection. For a single product-line firm, the payoffs from capital expenditures are apt to be related to each other mostly through a variable such as total sales. An extension of Sharpe's work to a number of uncorrelated indexes by K. J. Cohen [1] doubtlessly may be applied within this framework also.

C. Interdependent Investments with Probabilistic Returns

Taking cognizance of covariance between investment payoffs may be thought of as the simplest form of interdependence, and strict independence in our original sense does not apply. However, it is still necessary to consider such explicit interrelationships as mutual-exclusion and dependence in order to see what these concepts imply given that outcomes are random variables.

Continuing, then, with jointly normally distributed payoffs from all investments (thus narrowing our concern to mean, variance and covariance), we observe first that the choice among mutually-exclusive investments in the absence of budget or other constraints cannot be made without reference to the whole set of alternatives. That is, if the projects within set $\underline{J}$ are mutually-exclusive, so that acceptance of more than one of them can be ruled out in advance, the quantities $\underline{\mu}_j$ and $\underline{\sigma}_j$, $\underline{j} \in \underline{J}$, are insufficient for selection of the preferred one (if any). The covariances between the mutually-exclusive project and the inde-

[31] I.e., one with minimum variance for given mean, or maximum mean for given variance.

pendent ones enter into the choice, which may be formulated once more as a quadratic integer program solved by use of Reiter's method:

$$\text{V.11} \quad B = \begin{pmatrix} \mu_1 - \sigma_1^2 & -M & -2\,\sigma_{13} & \cdots & 2\,\sigma_{1n} \\ 0 & \mu_2 - \sigma_2^2 & -2\,\sigma_{23} & \cdots & 2\,\sigma_{2n} \\ 0 & 0 & \mu_3 - \sigma_3^2 & \cdots & -2\,\sigma_{3n} \\ & \cdots & & \cdots & \\ 0 & 0 & 0 & \cdots & \mu_n - \sigma_n^2 \end{pmatrix}$$

In V.11, projects 1 and 2 are considered mutually-exclusive, introducing the penalty $-\underline{M}$ into the second column of the first row to prevent their joint acceptance. The matrix otherwise resembles V.2.[32]

Great care must be excercised in utilizing this approach for the generalized second order effect matrix, as in III.13. A difficulty arises around the meaning of the covariance terms, $\underline{\sigma_{ij}}$, when the off-diagonal elements include expected values. Consider, for example, a machine tool for which accessories are available which increase the range of products which the machine can produce, and at the same time, increase its reliability. We can consider the tool itself project $\underline{r}$, its accessories project $\underline{s}$, and thus denote the random payoff from the machine tool alone by $\underline{b_{rr}}$ with expectation $\underline{\mu_{rr}}$ and variance $\underline{\sigma_{rr}}$; the cost of the accessories by $\underline{b_{ss}} = \underline{\mu_{ss}}$, with $\underline{\sigma_{ss}} = 0$; and the increase in payoff due to the accessories by $\underline{b_{rs}}$ with expectation of $\underline{\mu_{rs}}$. However, we shall need $\underline{\sigma_{rs}}$ to denote more than the variance of $\underline{b_{rs}}$. If we define $\underline{\pi_{rs}} = \underline{b_{rr}} + \underline{b_{rs}} + \underline{b_{ss}}$, i.e., $\underline{\pi_{rs}}$ is the payoff from the compound projects, and $\underline{V_{rs}}$ is its variance, then

[32] There is obviously no need for $\underline{\sigma_{12}}$ since the $-\underline{M}$ will rule out $\underline{x_1^*} = 1$, $\underline{x_2^*} = 1$ from the optimal solution.

V.12 $$V_{rs} = \sigma_{rr} + \sigma_{ss} + \sigma_{rs} = \sigma_{rr} + \sigma_{rs}$$

or $$\sigma_{rs} = V_{rs} - \sigma_{rr} - \sigma_{ss} = V_{rs} - \sigma_{rr}.$$

With these definitions it is possible to treat complex interrelationships by use of Reiter's method, although the data preparation requires, in effect, computing the parameters of all possible (or likely) compound projects, which could then be handled via matrix V.11 just as well. Using this earlier method also has the advantage of preserving the meaning of the covariance term between independent projects which would be clouded when covariances between a dependent project and unrelated independent projects are needed.

A final note regarding the quantities subject to budget constraints is needed before this section may be concluded. We have assumed that these are known with certainty. Should they also be regarded as stochastic the character of the problem would change in a significant way. One alternative would be to make compliance with the constraint less rigid, e.g., by stating only a probability less than unity with which the condition must hold. This places the problem into the realm of chance constrainted programming [4a]. Another would be to proceed sequentially in the allocation process, to make certain that expenditures stay within the preset ceilings. The latter problem, which may be formulated as a dynamic programming problem, does take us outside the framework set for this paper and will be taken up elsewhere.

VI. R & D PROJECT SELECTION

A. Expected Value Maximization

The final section of this paper is devoted to consideration of an additional class of project interrelationships which can arise in the context of constructing a research and development program. In order to concentrate on those aspects we consider a drastically simplified problem in which the payoff from one successful development of a product or process is known with certainty

(again denoted by b_i), in which the development cost, c_i, is similarly known with certainty, and in which the probability of success is believed to be p_i for a single project i.

If projects are mutually independent in all respects, acceptance is contingent only on the condition

VI.1 $$p_i b_i - c_i > 0.$$

Where simultaneously resource limitations are also imposed, the problem has the format of the original Lorie-Savage problem, and the methods offered for this still apply. The project payoff is now given by the left side of VI.1. This much is of little interest. Consider, however, two projects, r and s which are mutually-exclusive in the following sense. They may represent alternative products to serve the same function, or products, e.g., chemicals or drugs, synthesized by two different methods. Then it may be that if research on both projects is undertaken and success is achieved on both, only the better project will actually be taken past the development stage. Thus, if $b_r > b_s$, and p_{rs} is the probability of success on both projects r and s, then the payoff from undertaking research on both is given by the applicable box in the following payoff matrix:

	Success on r	Failure on r
Success on s	$b_r - c_r - c_s$	$b_s - c_r - c_s$
Failure on s	$b_r - c_r - c_s$	$- c_r - c_s$

The expected payoff then is

VI.2 $$\pi_{rs} = p_r(b_r - c_r - c_s) + (p_s - p_{rs})\,(b_s - c_r - c_s) + (1 - p_r - p_s + p_{rs})(-c_r-c_s)$$
$$= p_r b_r + (p_s - p_{rs})\, b_s - (c_r + c_s)$$

and in the absence of budgets the decision to undertake both is based on a comparison of π_{rs} with the expected payoff on the better of the two projects alone, where "better" means a higher expected value of the undertaking. Thus, if

VI. 3 $$p_r b_r - c_r > p_s b_s - c_s$$

then $$\pi_{rs} > p_r b_r - c_r \quad \text{implies}$$

VI.4 $$(p_s - p_{rs})\, b_s - c_s > 0$$

or $$p_s - p_{rs} > \frac{c_s}{b_s}$$

and research on both should be undertaken. If the inequality VI.3 were reversed, even though $\underline{b_r} > \underline{b_s}$ as is assumed in VI.2, then the comparable criterion would require

VI.5 $$p_r b_r - c_r > p_{rs} b_s.$$

This argument may easily be generalized to an arbitrary number of alternatives. Where consideration of such mutually-exclusive alternatives is to be made alongside independent ones (to be discussed below) the matrix formulation of Reiter again suggests itself:

VI.6 $$B = \begin{pmatrix} p_r b_r - c_r & -p_{rs} b_s & \cdots\cdots \\ 0 & p_s b_s - c_s & \cdots\cdots \\ 0 & 0 & \cdots\cdots \end{pmatrix}$$

Should both projects $\underline{r}$ and $\underline{s}$ be selected the payoff using VI.6 is the same as in VI.2. In the matrix formulation it is possible to add an additional term, $\underline{c_{rs}}$ (which may be positive or negative), to express an adjustment to the total development costs resulting from the joint development program. This formulation also lends itself to the simultaneous consideration of three or more projects which are mutually-exclusive in this special sense. However, for every additional project we require an additional dimension in the array, the need for which arises

from a term involving $\underline{p}_{\underline{ijk\ldots s}}$, the joint probability of success on all such projects.

Additional interrelationships can arise when the probability of success on one project is affected by the undertaking of research on a non-competing project as when, for example, the research embodies similar approaches or instrumentation. These effects may be reciprocal, though not necessarily symmetric. A project $\underline{r}$ may involve a process jointly with $\underline{s}$, so that if both developments are undertaken the probability of success on each is enhanced. However, it may be that the joint process is only a small part of the problem involved in project $\underline{s}$ although it comprises the major component of project $\underline{r}$. Hence, if we denote by $\underline{p}_{\underline{rs}}$ the probability of success on project $\underline{r}$ given that project $\underline{s}$ is also undertaken[33] it may be that $\underline{p}_{\underline{rs}} \neq \underline{p}_{\underline{sr}}$. Hence we require a nonsymmetric matrix $\underline{P}$:

$$\text{VI.7} \qquad P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ & \cdots & & \cdots & \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn} \end{pmatrix}$$

where $\underline{p}_{\underline{ii}}$ represents the probability of success on project $\underline{i}$ taken by itself. This may be combined with a diagonal payoff matrix $\underline{B}$ and the triangular cost matrix $\underline{C}$, viz.,

$$\text{VI.8} \qquad B = \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ 0 & 0 & 0 & \vdots & b_n \end{pmatrix} \qquad C = \begin{pmatrix} c_1 & c_{12} & c_{13} & \cdots & c_{1n} \\ 0 & c_2 & c_{23} & \cdots & c_{2n} \\ 0 & 0 & c_3 & \cdots & c_{3n} \\ 0 & 0 & 0 & \vdots & c_n \end{pmatrix}$$

[33]This is a change in definition from that used above.

in the quadratic integer programming problem[34]

VI.9 $$\text{Maximize} \quad X[BP - C]X^t$$

where $\underline{X}$ is a vector of $\underline{x_i}$ as before. The quantity in brackets is a matrix which may be solved using Reiter's method. In addition, a budget constraint may be placed on the expenditures $\underline{c_{ij}}$ by utilizing a Lagrangian multiplier, as in Section IV, which may be represented by

VI.10 $$\text{Maximize} \quad X[BP - (1 - \lambda)C]X^t$$

for various values of $\underline{\lambda}$. We may substitute $\underline{\lambda}'$ for the term $(1 - \underline{\lambda})$ and solve

VI.11 $$\text{Maximize} \quad X[BP - \lambda' C]X^t.$$

where

VI.12 $$BP - \lambda' C = \begin{pmatrix} p_{11}b_1 - \lambda' c_1 & p_{12}b_1 - \lambda' c_{12} & p_{13}b_1 - \lambda' c_{13} & \cdots \\ p_{21}b_2 & p_{22}b_2 - \lambda' c_2 & p_{23}b_2 - \lambda' c_{23} & \cdots \\ p_{31}b_3 & p_{32}b_3 & p_{33}b_3 - \lambda' c_3 & \cdots \\ & & \cdots & \cdots \end{pmatrix}$$

Finally, the methods for handling mutually-exclusive projects and mutually independent projects may be combined. Some of the columns of VI.12 would then be as given in matrix VI.6, with $\underline{p_{rs}}$ as defined there, and with terms below the diagonal as defined for VI.12.

B. Nonlinear Criterion Functions

Nonlinear utility or criterion functions give rise to a host of considerations not present when expected value maximization serves as the criterion. A brief discussion of some of the issues in the context of R & D project selection will conclude this presentation. As before, we continue to regard the objective

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[34] The matrix $\underline{C}$ is still triangular since terms $\underline{c_{ij}}$ enter only once.

of the decision maker to select the set of projects which maximizes expected utility, but the utility function is not linear in the outcomes. For simplicity we shall illustrate our remarks by reference to a quadratic utility function,

VI.13 $$U(y) = y - \alpha y^2$$

recognizing that his criterion is increasingly risk averse in Pratt's sense [23]. For present purposes, however, it is adequate. (α here is the coefficient of risk aversion).

The first observation to be made is that the assumption of a static decision procedure, i.e., one which fails to take into account other decisions which have been made in the past and will be made in the future, is no longer tenable. Utility is defined for the entire "portfolio" of projects, including both the project being considered and those already accepted and in process of being carried out. Consider, under perfect certainty, an existing portfolio of value y and two potential projects, worth z_1 and z_2. From VI.13 we may express the utility of a sum of two outcomes, $y + z$, by

VI.14 $$U(y + z) = U(y) + U(z) - 2\alpha yz$$

so that, given y, the requirement for acceptance of z is an increase in total utility. Since y is not subject to reduction (unlike the securities portfolio of Markowitz) the condition for acceptability reduces to

VI.15 $$U(z) > 2\alpha yz$$

With two potential independent projects with payoffs of z_1 and z_2, the outcome of a decision procedure based on criterion VI.15 alone could depend on the order in which the projects were considered. If project one were taken up first, acceptance of both would require

VI.16 a) $U(z_1) > 2\alpha z_1 y$

b) $U(z_2) > 2\alpha z_2\ (y + z_1) = 2\alpha z_2 y + 2\alpha z_1 z_2$

if the first had been accepted. Alternatively, taking up the decision about the second project first requires

VI.17 a) $U(z_2) > 2\alpha z_2 y$

b) $U(z_1) > 2\alpha z_1\ (y + z_2) = 2\alpha z_1 y + 2\alpha z_1 z_2$

if the second project had been accepted. Suppose that both projects would be acceptable if they were the only ones being considered, i.e.,

VI.18 a) $U(z_1) > 2\alpha z_1 y$

b) $U(z_2) > 2\alpha z_2 y$

but that while

VI.19 $U(z_1) > 2\alpha z_1\ (y + z_2)$

also

VI.20 $U(z_2) < 2\alpha z_2\ (y + z_1)$.

Given such values of $\underline{z_1}$, $\underline{z_2}$ and $\underline{y}$, the second project would be accepted only if it were taken up first, while this restriction would not apply to the first project.

It is clearly undesirable for a decision procedure to depend on the order in which projects are taken up. Yet, this would be the effect with naive application of a nonlinear utility function.[35] While it is true that both projects $\underline{r}$ and $\underline{s}$ would be accepted given the condition of accepting both or neither,[36] maximization of expected utility would lead to acceptance of project $\underline{r}$ alone. This

[35] Although these illustrations make use of a quadratic utility function, the con clusion applies to a large variety of nonlinear criterion functions, in particular also to ones which are decreasingly risk averse.

[36] $U(y + z_1+z_2) > U(y)$.

may be seen by regarding VI.19 as an equation, and rewriting VI.18b as

VI.21 $$U(z_2) = 2\alpha z_2 y + \varepsilon, \qquad \varepsilon < 2\alpha z_1 z_2.$$

From these one may derive

VI.22
$$\text{a)}\quad U(y + z_1) \equiv U(y) + U(z_1) - 2\alpha y z_1 = U(y) + 2\alpha z_1 z_2$$
$$\text{b)}\quad U(y + z_2) \equiv U(y) + U(z_2) - 2\alpha y z_2 = U(y) + \varepsilon < U(y) + 2\alpha z_1 z_2$$
$$\text{c)}\quad U(y + z_1 + z_2) \equiv U(y) + U(z_1) + U(z_2) - 2\alpha(yz_1 + yz_2 + z_1 z_2) = U(y) + \varepsilon < U(y) + 2\alpha z_1 z_2$$

leading to the stated conclusion.

An additional complication when the outcomes are random variables (as they are in the present context), is that $\underline{y}$ itself, the value of the previously accepted portfolio, is also random. This requires that the joint distribution of the outcomes of present and potential projects must enter the calculations.

In the strictly static case in which selection is to be made so as to maximize the utility of the projects being chosen without regard to past selections, we first derive the criterion for the Markowitz-type of R & D project portfolio consisting only of projects which are not interrelated in the physical sense. Statistical independence, however, is not assumed. We begin with two independent projects, and generalize from there. Given two projects $\underline{r}$ and $\underline{s}$ with development costs (negative revenues) of $(-\underline{c_r})$ and $(-\underline{c_s})$, with net present values after successful development of $\underline{b_r}$ and $\underline{c_r}$, and with probabilities of $\underline{p_{rs}}$ of joint success, $(\underline{p_r}-\underline{p_{rs}})$ and $(\underline{p_s}-\underline{p_{rs}})$ of success on $\underline{r}$ alone and $\underline{s}$ alone, respectively, expected utility from acceptance of both projects, $\underline{U_{rs}}$, is given by

VI.23 $$U_{rs} = p_{rs}\, U(b_r - c_r + b_s - c_s) + (p_r - p_{rs})\, U(b_r - c_r - c_s) + (p_s - p_{rs})\, U(b_s - c_s - c_r) + (1 - p_r - p_s + p_{rs})\, U(-c_r - c_s)$$

which simplifies to

VI.24 $$U_{rs} = U_r + U_s - 2\alpha\, E_{rs}$$

where

VI.25 $$U_j = p_j\, U(b_j - c_j) + (1 - p_j)\, U(-\, c_j)$$

and[37]

VI.26 $$E_{ij} = p_{ij}b_ib_j - p_ib_ic_j - p_jb_jc_i + c_ic_j.$$

Expression VI.24 may be generalized to any number of independent projects, so that mazimization of expected utility from acceptance of a set of projects may be written in terms parallel to model V.1, viz.,

VI.27 $$\text{Maximize} \quad \sum_{i=1}^{n} U_ix_i - 2\alpha \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_iE_{ij}x_j$$

$$\text{subject to} \quad x_i = \begin{cases} 0 \\ 1 \end{cases}, \; i=1,\ldots,n$$

which is once more in a form suitable for solution by Reiter's method, with payoff matrix

VI.28 $$B = \begin{pmatrix} U_1 & -2\alpha E_{12} & -2\alpha E_{13} & \cdots & -2\alpha E_{1n} \\ 0 & U_2 & -2\alpha E_{23} & \cdots & -2\alpha E_{2n} \\ 0 & 0 & U_3 & \cdots & -2\alpha E_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & U_n \end{pmatrix}.$$

A budget constraint may be appended here by use of a Lagrangian multiplier.

Unfortunately, the relatively simple structure of VI.28 is lost in the face of any number of the project interrelationships already discussed. We single out one of these, the situation in which if parallel research is under-

[37]Note that the term $\underline{E_{ij}}$ is simply the product of expected outcomes, $(\underline{p_ib_i} - \underline{c_i})\ (\underline{p_jb_j} - \underline{c_j})$ with the term $\underline{p_ip_jb_ib_j}$ replaced by $\underline{p_{ij}b_ib_j}$.

taken and success is achieved on both lines of attack only the project with higher expected payoff[38] is actually carried out. For two such projects, r and s, the expression for expected utility corresponding to VI.23 is

$$\text{VI.29} \qquad U_{rs} = p_r\, U(b_r - c_r - c_s) + (p_s - p_{rs})\, U(b_s - c_s - c_r) + (1 - p_r - p_s + p_{rs})\, U(-c_r - c_s)$$

which reduces to

$$\text{VI.30} \qquad U_{rs} = U_r + U_s - 2\alpha E_{rs} - p_{rs}\,[U(b_s) - 2\alpha b_s(b_r - c_r - c_s)].$$

Adding an independent project, v, however, requires introduction of terms involving p_{rsv}, the probability of joint success on projects r, s and v into U_{rsv}, hence losing the important property of separability, which has been the chief characteristic of our models of capital budgeting of interrelated projects.

[38] This quantity strictly should be expected utility. However, we can make our point without opening a pandora's box of problems of evaluating the utility of future events today, expecially without complete knowledge of the future outcomes which will obtain then.

APPENDIX A FLOW CHART OF THE DYNAMIC PROGRAMMING CODE FOR CAPITAL BUDGETING[39]

The program flow-charted was written in FORTRAN with the exception of the strategy vectors and their manipulation. Since these consist of strings of zeros and ones, memory space and time was conserved by programming these in binary arithmetic. The program as written allows for a maximum of 500 strategies at each stage and is able to handle twenty separate constraints. Cord's problem [8, p. 340] involving 25 projects was solved in 63 seconds on an IBM 7094. For a rough estimate, this compares with 12 minutes reported by Cord in his application of the Lagrangian multiplier technique, but using an IBM 7070. As pointed out in the text, Cord failed to obtain the optimum, and the time to reach it would have necessitated a minimum of 14% additional time. The program was also tried out on a number of interdependencies, the total number of constraints actually utilized in any one problem was six. Three additional constraints in Cord's problem increased the time utilized to 90 seconds.

The definitions used in the flow chart are as follows:

NN = No. of projects in problem LM = No. of constraints in problem

N(I) = No. of strategies at stage I M = Maximum number of strategies permitted

b_i = payoff on project i C_j = amount of resource of type j available

c_{ij} = amount of resource of type j used by project i

$b = (b_1, b_2, \ldots b_{NN})$ $c_j = (c_{1j}, c_{2j}, \ldots, c_{NN,j})$ $C = (0, C_1, C_2, \ldots, C_{LM})$

$P(I) = (b_i, -c_{i1}, -c_{i2}, \ldots, -c_{i,LM})$

$R(I) = (b_i, C_1 - c_{i1}, C_2 - c_{i2}, \ldots, C_{LM} - c_{i,LM})$

$X(I) = (0,0,\ldots,0,1,0,\ldots,0)$ with 1 in the i^{th} position

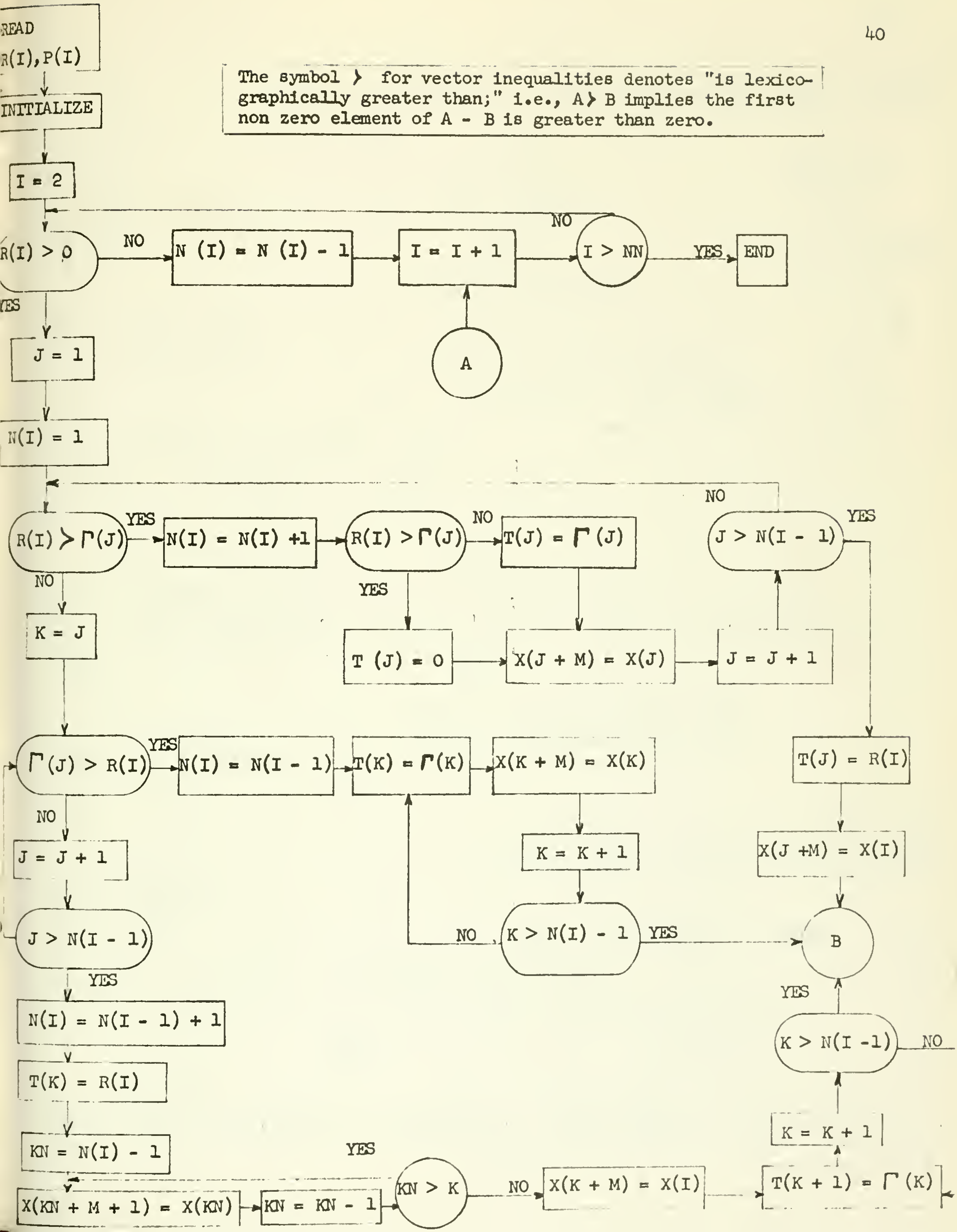
$X(J) = j^{th}$ strategy, i.e., a vector of 0's and 1's with 1 in the i^{th} position if $x_i = 1$

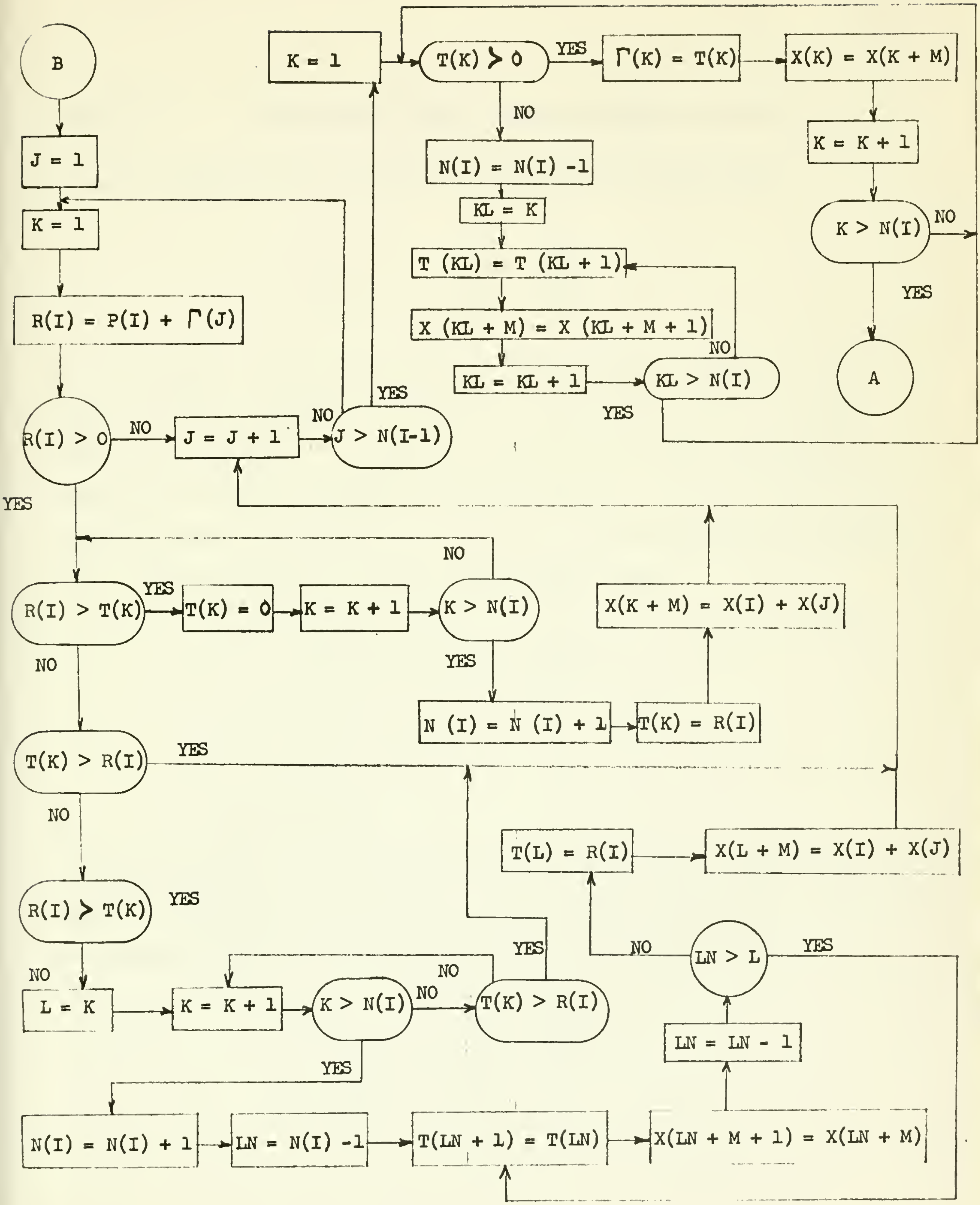
$\Gamma(J), T(J) = (C^t - (-b, c_1, c_2 \ldots, c_{LM})^t [X(J)]^t)^t$, a vector of payoffs and unused resources of the j^{th} strategy. T(J) is a temporary list, (J) is the revised list.

---

[39] The program was written by Stanley Sachar and David Ness.

The symbol $\succ$ for vector inequalities denotes "is lexicographically greater than;" i.e., $A \succ B$ implies the first non zero element of A - B is greater than zero.




B
J = 1
K = 1
R(I) = P(I) + Γ(J)
R(I) > O
NO
J = J + 1
NO
J > N(I-1)
YES
K = 1
T(K) > O
YES
Γ(K) = T(K)
X(K) = X(K + M)
K = K + 1
K > N(I)
NO
YES
A
NO
N(I) = N(I) -1
KL = K
T (KL) = T (KL + 1)
X (KL + M) = X (KL + M + 1)
NO
KL = KL + 1
KL > N(I)
YES
YES
R(I) > T(K)
YES
T(K) = O
K = K + 1
K > N(I)
NO
YES
N (I) = N (I) + 1
T(K) = R(I)
X(K + M) = X(I) + X(J)
NO
T(K) > R(I)
YES
NO
R(I) > T(K)
YES
NO
L = K
K = K + 1
K > N(I)
NO
NO
T(K) > R(I)
YES
YES
T(L) = R(I)
X(L + M) = X(I) + X(J)
NO
LN > L
YES
LN = LN - 1
N(I) = N(I) + 1
LN = N(I) -1
T(LN + 1) = T(LN)
X(LN + M + 1) = X(LN + M)

APPENDIX B EXAMPLE USING REITER'S DISCRETE OPTIMIZING METHOD

We seek the optimal list of projects for the following payoff matrix

$$B = \begin{pmatrix} 3 & 0 & 2 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{B.1}$$

Let the (randomly or otherwise selected) starting in-list $\alpha = \{1,3,4\}$; i.e., we begin with a list consisting of projects 1, 3 and 4. Crossing out row 2 and column 2, we add the payoff elements not crossed out: $3+2+(-1)+(-1)+(-3)+4 = 4$. Therefore the payoff corresponding to this in-list α is $\alpha = 4$. In the table we give the payoffs and gradients $G_{\alpha'} = \Gamma_{\alpha'} - \Gamma_{\alpha}$ for all in-list α' "connected" to α -- lists which differ in only a single project index.

α'	$\Gamma_{\alpha'}$	$G_{\alpha'}$
1,3	4	0
1,4	6	2
3,4	0	-4
→ 1,2,3,4	7	3

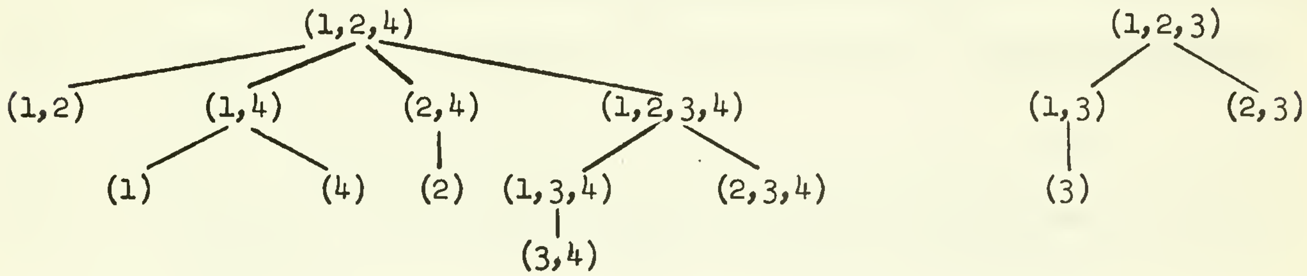
The largest improvement resulting from dropping one project or adding one to the in-list $\alpha = \{1,3,4\}$ is 3, which is associated with $\alpha' = \{1,2,3,4\}$. This forms the next starting in-list, which is once again evaluated in the table.

α'	α'	$G_{\alpha'}$
1,2,3	7	0
→ 1,2,4	8	1
1,3,4	4	-3

Here the largest improvement is associated with the move to $\alpha' = \{1,2,4\}$, yielding a total payoff of 8. It may be seen to be a local maximum from the next table, which takes this in-list as the starting point. Only negative values for $G_{\alpha'}$ result.

α'	$\Gamma_{\alpha'}$	$G_{\alpha'}$
1,2	5	-3
1,4	6	-2
1,2,3,4	7	-1

The local optimal list $\alpha = \{1,2,4\}$ turns out to be the global optimum. The tree-structure of this example is



The other local optimum consists of projects 1,2,3 with a payoff of 7. Since the 15 lists are divided into two groups of 11 on the tree of the global optimum list and 4 on the local non-global optimum list, the prior adds that a random sample leads to the global optimum here are $\frac{11}{15} = .73$. For further discussion see [26]

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